

TENSORS

We will use the tensorial calculus only to simplify the presentation of the equations. None of their properties will be used. This justifies, in this review, the absence the mathematical definition of the tensors which we will consider as a square array of nine elements:

$$\bar{T} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} = (t_{ij}) \quad (1)$$

i and j show respectively rows and columns.

The sum of two tensors:

$$\begin{aligned} \bar{T} + \bar{J} &= \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} + \begin{pmatrix} s_{11} & s_{22} & s_{33} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} = t_{ij} + s_{ij} \\ &= \begin{pmatrix} t_{11} + s_{11} & t_{21} + s_{21} & t_{31} + s_{31} \\ t_{21} + s_{21} & t_{22} + s_{22} & t_{23} + s_{23} \\ t_{33} + s_{33} & t_{32} + s_{32} & t_{33} + s_{33} \end{pmatrix}. \end{aligned} \quad (2)$$

Unit tensor:

$$\bar{I} = (\delta_{ij}) \quad \text{with} \quad \begin{aligned} \delta_{ij} &= 0 & \text{if} & \quad i \neq j \\ \delta_{ij} &= 1 & \text{if} & \quad i = j \end{aligned}$$

therefore:

$$\bar{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3)$$

Tensorial product of two vectors:

$$\vec{V} = (v_i) \quad \vec{W} = (w_j) \quad \text{with:} \quad i = 1, 2, 3 \quad j = 1, 2, 3$$

therefore:

$$\vec{W} = \vec{V} \vec{W} = (v_i w_j)$$

or

$$\vec{T} = \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{pmatrix}. \quad (4)$$

This product is also called the diadic product of two vectors.

Vectorial product of a tensor and a vector

a) Right product:

$$\vec{T} \cdot \vec{V} = (t_{ik} \cdot v_k) = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} t_{11}v_1 + t_{12}v_2 + t_{13}v_3 \\ t_{21}v_1 + t_{22}v_2 + t_{23}v_3 \\ t_{31}v_1 + t_{32}v_2 + t_{33}v_3 \end{pmatrix} \quad (5)$$

b) Left product

$$\vec{V} \cdot \vec{T} = (v_k \cdot t_{ki}) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} = \begin{pmatrix} v_1 t_{11} + v_2 t_{21} + v_3 t_{31} \\ v_1 t_{12} + v_2 t_{22} + v_3 t_{32} \\ v_1 t_{13} + v_2 t_{23} + v_3 t_{33} \end{pmatrix} \quad (6)$$

The product of a vector and a tensor is a vector. In the above summary, we used the convention of mute index which means that when an index is repeated twice in a monome a summation should be carried out it. If a tensor is symmetrical with respect to its principal diagonal, the left and right products of this tensor with the same vector are equal.

Divergence vector:

The divergence vector is defined as:

$$\vec{\nabla} = \frac{\partial}{\partial x_i} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \quad (7)$$

Vectorial divergence of a tensor:

The vectorial divergence of a vector id given by:

$$\vec{\nabla} \cdot \vec{T} = (\nabla_k \cdot t_{ki})$$

or

$$\vec{\nabla} \cdot \bar{T} = \begin{pmatrix} \frac{\partial t_{11}}{\partial x} + \frac{\partial t_{12}}{\partial y} + \frac{\partial t_{13}}{\partial z} \\ \frac{\partial t_{12}}{\partial x} + \frac{\partial t_{22}}{\partial y} + \frac{\partial t_{32}}{\partial z} \\ \frac{\partial t_{13}}{\partial x} + \frac{\partial t_{23}}{\partial y} + \frac{\partial t_{33}}{\partial z} \end{pmatrix} \quad (8)$$

Tensorial gradient of a vector:

Tensorial gradient of a vector is given by:

$$\vec{\nabla} \vec{V} = (\nabla_i V_j) = \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_2}{\partial x} & \frac{\partial v_3}{\partial x} \\ \frac{\partial v_1}{\partial y} & \frac{\partial v_2}{\partial y} & \frac{\partial v_3}{\partial y} \\ \frac{\partial v_1}{\partial z} & \frac{\partial v_2}{\partial z} & \frac{\partial v_3}{\partial z} \end{pmatrix} \quad (9)$$

Scalar product of two tensors:

The scalar product of two tensor is given by:

$$\bar{T} : \bar{S} = t_{11}s_{11} + t_{12}s_{12} + t_{13}s_{13} + t_{21}s_{21} + t_{22}s_{22} + t_{23}s_{23} + t_{31}s_{31} + t_{32}s_{32} + t_{33}s_{33} \quad (10)$$